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COMPLETE EXPRESSIONS FOR THE GRAVITATIONAL AND BUOYANCY FORCE TERMS IN THE EQUATIONS OF MOTION OF A SUBMERGED BODY

by

Frederick H. Imlay

HYDROMECHANICS LABORATORY

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NOTATION

B	Magnitude of $\overline{B}$
$\overline{B}$	Resultant force caused by static pressure of fluid acting on boundary of body
CB	Center of bouyancy (the point where $\overline{B}$ appears to act)
CG	Center of mass (the point where $\overline{W}$ appears to act)
K, M, N	Components of resultant moment of $\overline{B}$ and $\overline{W}$ about O in the directions of the x, y, z axes
K_B, M_B, N_B	Moments of $\overline{B}$ about the x, y, z axes
K_G, M_G, N_G	Moments of $\overline{W}$ about the x, y, z axes
O	Origin for the x, y, z axes
$\overline{Q}$	Combined moment of $\overline{B}$ and $\overline{W}$ about O
$\overline{Q}_B$	Moment of $\overline{B}$ about O
$\overline{Q}_G$	Moment of $\overline{W}$ about O
$\overline{r}_B$	Position vector of CB with respect to O
$\overline{r}_G$	Position vector of CG with respect to O
W	Magnitude of $\overline{W}$
$\overline{W}$	Resultant force caused by gravitational attraction on mass of body. Water inside the boundary of the body, whether in hard tanks or free-flooding, normally is considered part of the mass.
$X, Y, Z,$	Components of the resultant of $\overline{B}$ and $\overline{W}$ in the directions of the x, y, z axes
X_θ, X_ϕ, etc.	Partial derivatives $\frac{\partial X}{\partial \theta}, \frac{\partial X}{\partial \phi}$, etc.
X_0	Initial value of X force
x, y, z	Coordinate axes fixed in the body (See DTMB Report 1319[1])
x_B, y_B, z_B	Coordinates of r_B in the x, y, z axes

[1] References are listed on page 9.

x_G, y_G, z_G	Coordinates of $\bar{r}_G$ in the x, y, z axes
x_I, y_I, z_I	A set of orthogonal, right-hand, Cartesian coordinate axes fixed in space (The z_I axis is assumed fixed in the direction of gravitational attraction.)
Δ	Prefix indicating an incremental change in a quantity
ψ, θ, ϕ	Euler angles giving the orientation of the x, y, z axes with respect to the x_I, y_I, z_I axes (See Reference 1)
ψ_0, θ_0, ϕ_0	Initial values of ψ, θ, ϕ

ABSTRACT

Complete expressions are given for the forces and moments acting on a submerged body because of gravitational attraction on the mass of the body and buoyancy force acting on the body boundary. The variations of the forces and moments with changes in angular attitude of the body are discussed in detail. The relationship of these variations to the phenomenon of metacentric stability is described.

INTRODUCTION

Frequently the motion of an underwater body is treated by considering that the body is ballasted until it is in neutral buoyancy, that is, the mass of the body is adjusted to equal the displacement of the body. Coupled with such an approach, it is usually assumed that the location of the center of mass simultaneously is adjusted to a position directly under the center of buoyancy. Such a body is said to be in neutral buoyancy and level trim.

No great complication is introduced in the equations of motion if one assumes that the body is neither in level trim nor in neutral buoyancy—in fact, such an assumption is usually more representative of the true condition, an out-of-trim state either by design or because of leakage. As an interesting aside, leakage may be of air at a ballast tank blow valve, thus causing the body to be "light."

The treatments of the out-of-trim motion of an underwater body which exist in the literature usually employ expressions which have been "linerarized" to some degree. The author felt that complete, nonlinear, expressions for the gravitational and buoyancy terms might be useful because such expressions would be exact regardless of the magnitude of the angular departure from "even keel." Furthermore the proposed expressions would involve no restrictions as to the locations of the center of buoyancy and center of mass with respect to the origin of the body axes. Such exact expressions can readily be linearized to the extent warranted by a particular problem. The simplification of the basic expressions in the case of a plane of symmetry is illustrated.

The material reported here is part of a broader study of the general equations of motion of a body in a fluid being carried out under the General Hydromechanics Research Program at the David Taylor Model Basin. Other related work is contained in References 2 and 3.

GENERAL CONSIDERATIONS

The phenomenon described in the field of naval architecture either as static stability or, preferably, as metacentric stability results from the interaction of two external forces acting on a body that has part or all of its outer envelope in contact with a fluid. These forces are called the gravitational force and buoyancy force. The importance of metacentric stability in the dynamics of underwater bodies has been recognized for some time.[4] The equally important metacentric stability of a surface ship is a slightly more involved concept and lies outside the scope of this paper.

BUOYANCY FORCE

The buoyancy force frequently is called displacement. It is a function of only the form of the outer envelope of the body and the presence of a static pressure gradient in the fluid. The static pressure gradient exists only because of the gravitational attraction on the mass elements of the fluid.

Surfaces of equal static pressure may be visualized as horizontal laminations in the fluid, with the pressure increasing as one goes deeper because each lamination is supporting the weight of all the layers of fluid above it. Thus the direction of maximum static pressure gradient is vertically down.

Any body immersed in the fluid will have some portion that penetrates into the deeper laminations and hence will experience a higher static pressure on this lower portion. It follows that the body is acted upon by a net buoyancy force upward, with a line of action directly opposite to the direction of gravitational attraction.

If the fluid and the body are incompressible, the magnitude of the buoyancy force is directly proportional to the volume of fluid displaced by the body and is independent of the angular attitude or depth of the body. Conceptually, the buoyancy force may then be considered to act at a fixed point whose location is called the center of buoyancy. The position of the center of buoyancy is governed solely by the shape of the body-fluid boundary.

Generally speaking, hull compressibility will not cause significant migrations of the center of buoyancy of an underwater body but may account for sensible changes in the magnitude of the buoyancy force. The compressibility of water is considered of no practical importance unless very great depth changes are envisioned; density is of the order of 5 percent higher in the greatest ocean depths (35,000 feet) than on the surface. The "density layers" caused by abrupt changes in salinity or temperature normally are much more significant.

GRAVITATIONAL FORCE

Gravitational attraction acts on all the mass elements of the body. The distribution of mass within the body determines the location of a point called the center of mass, about which the body will balance in a uniform gravitational field. Strictly speaking, the integrated effect of gravitational attraction on all the mass elements of the body gives rise to a net gravitational force which may be assumed to act at a point called the center of gravity. The center of gravity will differ from the center of mass only if the gravitational field is nonuniform.

In relation to the probable size of underwater bodies, the earth's gravitational field may be considered uniform; furthermore, the center of mass is of special importance if accelerated motions of the body occur, hence the preference for that term, vis-a-vis center of gravity, in discussions of dynamics.

In typical underwater bodies, the magnitude of the gravitational force and the location of its point of action are adjustable by ballasting; i.e., the mass and center of mass are variable.

METACENTRIC STABILITY

The existence of metacentric stability depends upon the failure of the center of mass and the center of buoyancy to coincide. Assume a body submerged in water, motionless, and on "even keel." Furthermore, assume that the body has been ballasted so that the center of mass is directly below the center of buoyancy and that the mass has been adjusted to produce a gravity force (hereafter called weight) just equal to the buoyancy force. Such a body will remain motionless at its present depth. If the body then is disturbed in either pitch or roll (and the disturbing cause subsequently removed), the body will return to its initial attitude. Such a body is said to possess the characteristic of metacentric stability. The stability exists because any angular displacement about any axis (except the vertical line containing the center of mass and center of buoyancy) shifts the lines of action of the two forces to two parallel locations such that a couple is produced tending to decrease the angular displacement.

Two important variations on the preceding situation can be pictured In the first, the body is in neutral buoyancy but not in level trim. As a consequence, the center of buoyancy is not above the center of mass. When the body is released in the stated initial conditions, it will rotate to a new attitude where the two centers do lie on a common vertical line; in the new attitude, the configuration will have metacentric stability, as described before. Note that a net initial moment exists which tends to rotate the body to its new attitude.

In the second important variation, the center of buoyancy is above the center of mass but the weight and buoyancy are not equal. There is a net initial vertical force present, therefore, which will accelerate the body when it is released under the stipulated initial conditions. For discussion, assume that the weight is greater than the buoyancy. Then a fraction of the weight equal to the buoyancy can be paired with the buoyancy to create the conditions for metacentric stability described in the first paragraph of this section. The remainder of the weight constitutes an accelerating force at the center of mass which can exist simultaneously with the conditions for metacentric stability.

A third variation that combines the cases described in the two preceding paragraphs does not introduce any new concepts.

SUMMARIZING REMARKS

The preceding discussion has sought to develop two principal points in the general case of a submerged body not in neutral buoyancy, namely: (1) there is an initial net vertical force and a net moment, at a given initial attitude, and (2) with changes in angular attitude there are changes in the moment, represented by the phenomenon of metacentric stability. Because a net vertical force exists, its presence leads to force changes along the body axes if the angular attitude changes.

DERIVATION OF GENERAL EXPRESSIONS

In this section, the complete expressions will be given first for the force and moment on an out-of-trim submerged body. A presentation of what are called "attitude derivatives" will then be given. These derivatives describe the rate-of-change of force and moment with change in angular attitude.

EXPRESSIONS FOR FORCE AND MOMENT

Assume a set of orthogonal, right-hand, Cartesian coordinate axes x_I, y_I, z_I fixed in space, with the z_I axis aligned in the direction of gravitational attraction. This set will be called inertial axes. Also assume a submerged body with a similar set of axes x, y, z fixed in the body at the point O. Let ψ, θ, ϕ be Euler angles fixing the angular orientation x, y, z with respect to x_I, y_I, z_I (see Reference 1).

Let $\overline{B}$ be a vector representing the buoyancy force on the body, and $\overline{W}$ a vector representing the force of gravitational attraction on the mass of the body. Magnitudes of the vectors are represented by omitting the bars over the letters. Then

$$W = mg \qquad [1]$$

where m is the mass of the body and g is the acceleration of gravity.

Consider now the resultant of the buoyancy and weight forces, the vector sum of $\overline{W}$ and $\overline{B}$. The magnitude of the resultant is $W - B$ because $\overline{B}$ is opposite in direction to $\overline{W}$. The selection for the orientation of the inertial axes was such that the resultant has a component only along the z_I axis, and its amount is $W - B$. The resultant also has components along the x, y, z axes, which are represented by X, Y, Z, respectively.

The following table of direction cosines can be used to obtain the components X, Y, Z in terms of the components that the resultant has in the inertial axes.

	X	Y	Z
0	$\cos\psi\cos\theta$	$-\sin\psi\cos\phi+\cos\psi\sin\theta\sin\phi$	$\cos\psi\sin\theta\cos\phi+\sin\psi\sin\phi$
0	$\sin\psi\cos\theta$	$\cos\psi\cos\phi+\sin\psi\sin\theta\sin\phi$	$\sin\psi\sin\theta\cos\phi-\cos\psi\sin\phi$
$W-B$	$-\sin\theta$	$\cos\theta\sin\phi$	$\cos\theta\cos\phi$

Thus

$$\left.\begin{aligned} X &= -(W-B)\sin\theta \\ Y &= (W-B)\cos\theta\sin\phi \\ Z &= (W-B)\cos\theta\cos\phi \end{aligned}\right\} \qquad [2]$$

Assume that $\overline{B}$ has a point of action at the center of buoyancy CB. Let $\overline{r_B}$ be a vector giving the position of CB with respect to O. Similarly, let $\overline{W}$ act at the center of mass CG, and let $\overline{r_G}$ be its position vector. Then $\overline{B}$ has a moment about O, represented by the symbol $\overline{Q}_B$, and given by the vector cross product

$$\overline{Q_B} = \overline{r_B} \times \overline{B} \qquad [3]$$

Let $\overline{Q}_B$ have components K_B, M_B, N_B about the x, y, z axes, respectively. Then from Equation [3]

$$\left.\begin{aligned} K_B &= (-B\cos\theta\cos\phi)\,y_B + (B\cos\theta\sin\phi)\,z_B \\ M_B &= (B\sin\theta)\,z_B + (B\cos\theta\cos\phi)\,x_B \\ N_B &= (-B\cos\theta\sin\phi)\,x_B - (B\sin\theta)\,y_B \end{aligned}\right\} \qquad [4]$$

where x_B, y_B, z_B are components of $\overline{r_B}$ in the directions of the x, y, z axes.

The weight force vector $\overline{W}$ also has a moment about O according to

$$\overline{Q_G} = \overline{r_G} \times \overline{W} \qquad [5]$$

where $\overline{Q_G}$ is the moment about O.

Let $\overline{r_G}$ have components x_G, y_G, z_G in the directions of the x, y, z axes, and take K_G, M_G, N_G as components of $\overline{Q}_G$ about those axes. Then Equation [5] can be separated into

$$\left.\begin{aligned} K_G &= (W\cos\theta\cos\phi)\,y_G - (W\cos\theta\sin\phi)\,z_G \\ M_G &= (-W\sin\theta)\,z_G - (W\cos\theta\cos\phi)\,x_G \\ N_G &= (W\cos\theta\sin\phi)\,x_G + (W\sin\theta)\,y_G \end{aligned}\right\} \qquad [6]$$

Let $\overline{Q}$ be a vector representing the moment about O of the combined effects of $\overline{B}$ and $\overline{W}$, and let K, M, N be components of $\overline{Q}$ about the x, y, z axes. Then from Equations [4] and [6]

$$\left.\begin{aligned}
K &= \cos\theta \ \left[(y_G W - y_B B)\cos\phi - (z_G W - z_B B)\sin\phi\right] \\
M &= -(x_G W - x_B B)\cos\theta\cos\phi - (z_G W - z_B B)\sin\theta \\
N &= (x_G W - x_B B)\cos\theta\sin\phi + (y_G W - y_B B)\sin\theta
\end{aligned}\right\} \quad [7]$$

ATTITUDE DERIVATIVES

Assume that a submerged body is at some initial angular attitude specified by ψ_0, θ_0, ϕ_0. Then differentiation of Equations [2] and [7] with respect to θ or ϕ will supply the following series of complete expressions for the derivatives. (Note that Equations [2] and [7] are not functions of ψ.)

$$\left.\begin{aligned}
X_\theta &= -(W-B)\cos\theta_0 \\
X_\phi &= 0 \\
Y_\theta &= -(W-B)\sin\theta_0\sin\phi_0 \\
Y_\phi &= (W-B)\cos\theta_0\cos\phi_0 \\
Z_\theta &= -(W-B)\sin\theta_0\cos\phi_0 \\
Z_\phi &= -(W-B)\cos\theta_0\sin\phi_0 \\
K_\theta &= -\sin\theta_0 \ \left[(y_G W - y_B B)\cos\phi_0 - (z_G W - z_B B)\sin\phi_0\right] \\
K_\phi &= -\cos\theta_0 \ \left[(z_G W - z_B B)\cos\phi_0 + (y_G W - y_B B)\sin\phi_0\right] \\
M_\theta &= -(z_G W - z_B B)\cos\theta_0 + (x_G W - x_B B)\sin\theta_0\cos\phi_0 \\
M_\phi &= (x_G W - x_B B)\cos\theta_0\sin\phi_0 \\
N_\theta &= (y_G W - y_B B)\cos\theta_0 - (x_G W - x_B B)\sin\theta_0\sin\phi_0 \\
N_\phi &= (x_G W - x_B B)\cos\theta_0\cos\phi_0
\end{aligned}\right\} \quad [8]$$

Where $X_\theta = \dfrac{\partial X}{\partial \theta}$, $X_\phi = \dfrac{\partial X}{\partial \phi}$, etc.

EFFECTS OF LARGE PERTURBATIONS ON ATTITUDE DERIVATIVES

Note that Equations [8] are partial derivatives and, although they are valid for initial values ψ_0, θ_0, ϕ_0 however large, they apply only to small incremental changes $\Delta\theta$, $\Delta\phi$ from those initial conditions.

If the perturbations from the initial conditions are large, such that the approximations

$$\left.\begin{aligned} \sin \Delta\theta &\approx 0 \\ \cos \Delta\theta &\approx 1 \\ \sin \Delta\phi &\approx 0 \\ \cos \Delta\phi &\approx 1 \end{aligned}\right\} \qquad [9]$$

are not valid, Equations [8] must be replaced by more cumbersome expressions. For example, let X_0 be the initial value of X force due to an initial value of θ_0 so

$$X_0 = -(W-B) \sin \theta_0 \qquad [10]$$

from Equations [2]. After a large perturbation $\Delta\theta$, there will be an attendant change in X force of amount ΔX, whereby

$$X_0 + \Delta X = -(W-B) \sin (\theta_0 + \Delta\theta) = -(W-B)(\sin \theta_0 \cos \Delta\theta + \cos \theta_0 \sin \Delta\theta) \qquad [11]$$

Subtracting Equation [10] from Equation [11], the change in X force due to a change in θ is

$$\Delta X = -(W-B) \cos \theta_0 \sin \Delta\theta + (W-B) \sin \theta_0 (1 - \cos \Delta\theta) \qquad [12]$$

Note the comparison of Equation [12] with the first member of Equations [8].

A similar treatment can be given the remainder of Equations [2] and [7]. As an illustration

$$\begin{aligned} \Delta Y = & -(W-B) \sin \theta_0 \sin \phi_0 \sin \Delta\theta \cos \Delta\phi \\ & +(W-B) \cos \theta_0 \cos \phi_0 \cos \Delta\theta \sin \Delta\phi \\ & -(W-B) \sin \theta_0 \cos \phi_0 \sin \Delta\theta \sin \Delta\phi \\ & -(W-B) \cos \theta_0 \sin \phi_0 (1 - \cos \Delta\theta \cos \Delta\phi) \qquad [13] \end{aligned}$$

which may be compared with the third and fourth members of Equations [8]. Equations of the form of [12] and [13] hold for perturbations however large.

The perturbations may be moderately large where the approximations of Equations [9] do not hold but the assumptions can be made

$$\left.\begin{aligned} \sin \Delta\theta &\approx \Delta\theta \\ \cos \Delta\theta &\approx 1 \\ \sin \Delta\phi &\approx \Delta\phi \\ \cos \Delta\phi &\approx 1 \end{aligned}\right\} \qquad [14]$$

These approximations are valid to within 1 percent for angles up to 8.1 degrees. Then Equation [12] can be replaced by

$$\Delta X \approx -(W-B)\,(\cos\,\theta_0)\,\Delta\theta \qquad [15]$$

and Equation [13] by

$$\Delta Y \approx -(W-B)\,\left[(\sin\,\theta_0\,\sin\,\phi_0)\,\Delta\theta-(\cos\,\theta_0\,\cos\,\phi_0)\,\Delta\phi+(\sin\,\theta_0\,\cos\,\phi_0)\,\Delta\theta\Delta\phi\right] \qquad [16]$$

Note that $\Delta\theta\Delta\phi$ is less than 0.02, in radian measure, when the two perturbations are each 8 degrees. The term involving $\Delta\theta\Delta\phi$ in Equation [16], therefore, is of the same order as the term involving $1-\cos\,\Delta\theta\,\cos\,\Delta\phi$, which was dropped from Equation [13] as a result of applying Equations [14]. If the term involving $\Delta\theta\Delta\phi$ is also dropped, Equation [16] becomes

$$\Delta Y \approx -(W-B)\,\left[(\sin\,\theta_0\,\sin\,\phi_0)\,\Delta\theta-(\cos\,\theta_0\,\cos\,\phi_0)\,\Delta\phi\right] \qquad [17]$$

Comparison will show a mathematical equivalence of Equation [15] with the first two members, and of Equation [17] with the third and fourth members, of Equations [8]. The conclusion is made, therefore that Equations [8] are suitable for perturbations of up to 8 degrees in θ and ϕ from any arbitrary initial conditions of these quantities, provided that errors of less than 2 percent are acceptable. For larger excursions, or a more stringent accuracy requirement, expressions of the form of Equations [12] and [13] should be used.

EFFECT OF PLANE OF SYMMETRY ON ATTITUDE DERIVATIVES

If the body possesses a common plane of symmetry for both its geometric shape and its mass distribution, the attitude derivatives can be given in simpler form. Assume that the y axis is normal to the plane of symmetry and that the origin for the x, y, z axes lies in the plane of symmetry. Then

$$y_B = y_G = 0 \qquad [18]$$

and Equations [8] can be replaced by

$$\left.\begin{aligned} X_\theta &= -(W-B)\,\cos\,\theta_0 \\ X_\phi &= 0 \\ Y_\theta &= -(W-B)\,\sin\,\theta_0\,\sin\,\phi_0 \\ Y_\phi &= (W-B)\,\cos\,\theta_0\,\cos\,\phi_0 \end{aligned}\right\} \qquad [19]$$

$$
\left.
\begin{aligned}
Z_{\theta} &= -(W-B) \sin \theta_0 \cos \phi_0 \\
Z_{\phi} &= -(W-B) \cos \theta_0 \sin \phi_0 \\
K_{\theta} &= (z_G W - z_B B) \sin \theta_0 \sin \phi_0 \\
K_{\phi} &= -(z_G W - z_B B) \cos \theta_0 \cos \phi_0 \\
M_{\theta} &= -(z_G W - z_B B) \cos \theta_0 + (x_G W - x_B B) \sin \theta_0 \cos \phi_0 \\
M_{\phi} &= (x_G W - x_B B) \cos \theta_0 \sin \phi_0 \\
N_{\theta} &= -(x_G W - x_B B) \sin \theta_0 \sin \phi_0 \\
N_{\phi} &= (x_G W - x_B B) \cos \theta_0 \cos \phi_0
\end{aligned}
\right\} \quad [19]
$$

REFERENCES

1. Imlay, Frederick H., "A Nomenclature for Stability and Control," David Taylor Model Basin Report 1319 (May 1959).

2. Imlay, Frederick H., "The Complete Expressions for 'Added Mass' of a Rigid Body Moving in an Ideal Fluid," David Taylor Model Basin Report 1528 (Jul 1961).

3. Imlay, Frederick H., "The Path Angles and Angles of Attack and Drift for Large Motions of a Moving Body in a Moving Fluid," David Taylor Model Basin Report 1813 (May 1964).

4. Hoyt, E.D. and Imlay, F.H., "The Influence of Metacentric Stability on the Dynamic Longitudinal Stability of a Submarine," David Taylor Model Basin Report C-158 (Oct 1948).